SHARP BOUNDS IN TERMS OF THE POWER OF THE CONTRA-HARMONIC MEAN FOR NEUMAN-SÁNDOR MEAN

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Abstract. In the paper, the authors obtain sharp bounds in terms of the power of the contra-harmonic mean for Neuman-Sándor mean.

1. Introduction

For positive numbers $a, b > 0$ with $a \neq b$, the second Seiffert mean $T(a,b)$, the root-mean-square $S(a,b)$, Neuman-Sándor mean $M(a,b)$, and the contra-harmonic mean $C(a,b)$ are respectively defined in [9, 13] by

$$T(a,b) = \frac{a-b}{2\arctan[(a-b)/(a+b)]}, \qquad S(a,b) = \sqrt{\frac{a^2+b^2}{2}}, \tag{1.1}$$

$$M(a,b) = \frac{a-b}{2\operatorname{arcsinh}[(a-b)/(a+b)]}, \qquad C(a,b) = \frac{a^2+b^2}{a+b}. \tag{1.2}$$

It is well known [7, 8, 10] that the inequalities

$$M(a,b) < T(a,b) < S(a,b) < C(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

In [2, 3], the inequalities

$$S(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) < S(\beta a + (1-\beta)b, \beta b + (1-\beta)a) \tag{1.3}$$

and

$$C(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) < T(a,b) < C(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \tag{1.4}$$

were proved to be valid for $\frac{1}{2} < \alpha, \beta, \lambda, \mu < 1$ and for all $a, b > 0$ with $a \neq b$ if and only if

$$\begin{aligned} \alpha &\leq \frac{1}{2}\left(1 + \sqrt{\frac{16}{\pi^2} - 1}\right), \quad \beta \geq \frac{3+\sqrt{6}}{6}, \\ \lambda &\leq \frac{1}{2}\left(1 + \sqrt{\frac{4}{\pi} - 1}\right), \quad \mu \geq \frac{3+\sqrt{3}}{6} \end{aligned} \tag{1.5}$$

respectively. In [12], the double inequality

$$S(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < M(a,b) < S(\beta a + (1-\beta)b, \beta b + (1-\beta)a) \tag{1.6}$$

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was proved to be valid for $\frac{1}{2} < \alpha, \beta < 1$ and for all $a, b > 0$ with $a \neq b$ if and only if

$$\alpha \le \frac{1}{2}\left\{1+\sqrt{\frac{1}{\left[\ln\left(1+\sqrt{2}\,\right)\right]^2}-1}\right\} \quad \text{and} \quad \beta \ge \frac{3+\sqrt{3}}{6}. \tag{1.7}$$

For more information on this topic, please refer to recently published papers [4, 5, 6, 11] and references cited therein.

For $t \in \left(\frac{1}{2}, 1\right)$ and $p \ge \frac{1}{2}$, let

$$Q_{t,p}(a,b) = C^p(ta+(1-t)b, tb+(1-t)a)A^{1-p}(a,b), \tag{1.8}$$

where $A(a,b) = \frac{a+b}{2}$ is the classical arithmetic mean of a and b. Then, by definitions in (1.1) and (1.2), it is easy to see that

$$\begin{aligned} Q_{t,1/2}(a,b) &= S(ta+(1-t)b, tb+(1-t)a), \\ Q_{t,1}(a,b) &= C(ta+(1-t)b, tb+(1-t)a), \end{aligned}$$

and $Q_{t,p}(a,b)$ is strictly increasing with respect to $t \in \left(\frac{1}{2}, 1\right)$.

Motivating by results mentioned above, we naturally ask a question: What are the greatest value $t_1 = t_1(p)$ and the least value $t_2 = t_2(p)$ in $\left(\frac{1}{2}, 1\right)$ such that the double inequality

$$Q_{t_1,p}(a,b) < M(a,b) < Q_{t_2,p}(a,b) \tag{1.9}$$

holds for all $a, b > 0$ with $a \neq b$ and for all $p \ge \frac{1}{2}$?

The aim of this paper is to answer this question. The solution to this question may be stated as the following Theorem 1.1.

Theorem 1.1. *Let $t_1, t_2 \in \left(\frac{1}{2}, 1\right)$ and $p \in \left[\frac{1}{2}, \infty\right)$. Then the double inequality* (1.9) *holds for all $a, b > 0$ with $a \neq b$ if and only if*

$$t_1 \le \frac{1}{2}\left[1+\sqrt{\left(\frac{1}{t^*}\right)^{1/p}-1}\,\right] \quad \textit{and} \quad t_2 \ge \frac{1}{2}\left(1+\frac{1}{\sqrt{6p}}\right), \tag{1.10}$$

where

$$t^* = \ln\left(1+\sqrt{2}\,\right) = 0.88\dots. \tag{1.11}$$

Remark 1.1. When $p = \frac{1}{2}$ in Theorem 1.1, the double inequality (1.9) becomes (1.6).

Remark 1.2. If taking $p = 1$ in Theorem 1.1, we can conclude that the double inequality

$$C(\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a) < M(a,b) < C(\mu a+(1-\mu)b, \mu b+(1-\mu)a) \tag{1.12}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if

$$\frac{1}{2} < \lambda \le \frac{1}{2}\left[1+\sqrt{\frac{1}{\ln\left(1+\sqrt{2}\,\right)}-1}\,\right] \quad \text{and} \quad 1 > \mu \ge \frac{1}{2}\left(1+\frac{\sqrt{6}}{6}\right). \tag{1.13}$$

2. Lemmas

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 ([1, Theorem 1.25])**.** *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $g'(x) \ne 0$ and $\frac{f'(x)}{g'(x)}$ is strictly increasing (or strictly decreasing, respectively) on (a,b), so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}. \tag{2.1}$$

Lemma 2.2. *The function*

$$h(x)=\frac{\left(1+x^2\right)\operatorname{arcsinh} x}{x} \tag{2.2}$$

is strictly increasing and convex on $(0,\infty)$.

Proof. This follows from the following arguments:

$$h'(x)=\frac{x\sqrt{1+x^2}\,-\operatorname{arcsinh} x+x^2\operatorname{arcsinh} x}{x^2}\triangleq\frac{h_1(x)}{x^2},$$

$$h_1'(x)=x\left(\frac{3x}{\sqrt{1+x^2}}+2\operatorname{arcsinh} x\right)\triangleq xh_2(x),$$

$$h_2'(x)=\frac{5+2x^2}{(1+x^2)^{3/2}}>0$$

on $(0,\infty)$ and

$$\lim_{x\to0^+}h_1(x)=\lim_{x\to0^+}h_2(x)=0.$$

□

Lemma 2.3. *For $u\in[0,1]$ and $p\ge\frac12$, let*

$$f_{u,p}(x)=p\ln(1+ux^2)-\ln x+\ln\operatorname{arcsinh} x \tag{2.3}$$

on $(0,1)$. Then the function $f_{u,p}(x)$ is positive if and only if $6pu\ge1$ and it is negative if and only if $1+u\le\left(\frac{1}{t^*}\right)^{1/p}$, where t^* is defined by (1.11).

Proof. It is ready that

$$\lim_{x\to0^+}f_{u,p}(x)=0 \tag{2.4}$$

and

$$\lim_{x\to1^-}f_{u,p}(x)=p\ln(1+u)+\ln(t^*). \tag{2.5}$$

An easy computation yields

$$\begin{aligned}
f_{u,p}'(x)&=\frac{2pux}{1+ux^2}+\frac{1}{\sqrt{1+x^2}\,\operatorname{arcsinh} x}-\frac1x\\
&=\frac{u\left[(2p-1)x^2\sqrt{1+x^2}\,\operatorname{arcsinh} x+x^3\right]-\left[\sqrt{1+x^2}\,\operatorname{arcsinh} x-x\right]}{x(1+ux^2)\sqrt{1+x^2}\,\operatorname{arcsinh} x}\\
&=\frac{(2p-1)x^2\sqrt{1+x^2}\,\operatorname{arcsinh} x+x^3}{x(1+ux^2)\sqrt{1+x^2}\,\operatorname{arcsinh} x}\left[u-\frac{g_1(x)}{g_2(x)}\right],
\end{aligned} \tag{2.6}$$

where

$$g_1(x)=\operatorname{arcsinh} x-\frac{x}{\sqrt{1+x^2}} \quad\text{and}\quad g_2(x)=(2p-1)x^2\operatorname{arcsinh} x+\frac{x^3}{\sqrt{1+x^2}}.$$

Furthermore, we have

$$g_1(0) = g_2(0) = 0 \tag{2.7}$$

and

$$\frac{g_1'(x)}{g_2'(x)} = \frac{1}{2(2p-1)\sqrt{1+x^2}\,h(x) + (2p+1)x^2 + 2p + 2}, \tag{2.8}$$

where $h(x)$ is defined by (2.2). From Lemma 2.2, it follows that the quotient $\frac{g_1'(x)}{g_2'(x)}$ is strictly decreasing on $(0,1)$. Accordingly, from Lemma 2.1 and (2.7), it is deduced that the ratio $\frac{g_1(x)}{g_2(x)}$ is strictly decreasing on $(0,1)$.

Moreover, making use of L'Hôpital's rule leads to

$$\lim_{x\to 0} \frac{g_1(x)}{g_2(x)} = \frac{1}{6p} \tag{2.9}$$

and

$$\lim_{x\to 1} \frac{g_1(x)}{g_2(x)} = \frac{\sqrt{2}\,t^* - 1}{\sqrt{2}\,(2p-1)t^* + 1}. \tag{2.10}$$

When $u \ge \frac{1}{6p}$, combining (2.6) and (2.9) with the monotonicity of $\frac{g_1(x)}{g_2(x)}$ shows that the function $f_{u,p}(x)$ is strictly increasing on $(0,1)$. Therefore, the positivity of $f_{u,p}(x)$ on $(0,1)$ follows from (2.4) and the increasingly monotonicity of $f_{u,p}(x)$.

When $u \le \frac{\sqrt{2}\,t^*-1}{\sqrt{2}\,(2p-1)t^*+1}$, combining (2.6) and (2.10) with the monotonicity of $\frac{g_1(x)}{g_2(x)}$ reveals that the function $f_{u,p}(x)$ is strictly decreasing on $(0,1)$. Hence, the negativity of $f_{u,p}(x)$ on $(0,1)$ follows from (2.4) and the decreasingly monotonicity of $f_{u,p}(x)$.

When $\frac{\sqrt{2}\,t^*-1}{\sqrt{2}\,(2p-1)t^*+1} < u < \frac{1}{6p}$, from (2.6), (2.9), (2.10), and the monotonicity of the ratio $\frac{g_1(x)}{g_2(x)}$, we conclude that there exists a number $x_0 \in (0,1)$ such that $f_{u,p}(x)$ is strictly decreasing in $(0,x_0)$ and strictly increasing in $(x_0,1)$. Denote the limit in (2.5) by $h_p(u)$. Then, from the above arguments, it follows that

$$h_p\left(\frac{1}{6p}\right) = p\ln\left(1 + \frac{1}{6p}\right) + \ln(t^*) > 0 \tag{2.11}$$

and

$$h_p\left(\frac{\sqrt{2}\,t^* - 1}{\sqrt{2}\,(2p-1)t^* + 1}\right) = p\ln\left[1 + \frac{\sqrt{2}\,t^* - 1}{\sqrt{2}\,(2p-1)t^* + 1}\right] + \ln(t^*) < 0. \tag{2.12}$$

Since $h_p(u)$ is strictly increasing for $u > -1$, so it is also in $\bigl[\frac{\sqrt{2}\,t^*-1}{\sqrt{2}\,(2p-1)t^*+1}, \frac{1}{6p}\bigr]$. Thus, the inequalities in (2.11) and (2.12) imply that the function $h_p(u)$ has a unique zero point $u_0 = \bigl(\frac{1}{t^*}\bigr)^{1/p} - 1 \in \bigl(\frac{\sqrt{2}\,t^*-1}{\sqrt{2}\,(2p-1)t^*+1}, \frac{1}{6p}\bigr)$ such that $h_p(u) < 0$ for $u \in \bigl[\frac{\sqrt{2}\,t^*-1}{\sqrt{2}\,(2p-1)t^*+1}, u_0\bigr)$ and $h_p(u) > 0$ for $u \in \bigl(u_0, \frac{1}{6p}\bigr]$. As a result, combining (2.4) and (2.5) with the piecewise monotonicity of $f_{u,p}(x)$ reveals that $f_{u,p}(x) < 0$ for all $x \in (0,1)$ if and only if $\frac{\sqrt{2}\,t^*-1}{\sqrt{2}\,(2p-1)t^*+1} < u < u_0$. The proof of Lemma 2.3 is complete. $\square$

3. Proof of Theorem 1.1

Now we are in a position to prove our Theorem 1.1.

Since both $Q_{t,p}(a,b)$ and $M(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b$. Let $x=\frac{a-b}{a+b}\in(0,1)$. From (1.2) and (1.8), we obtain

$$\begin{aligned}\ln\frac{Q_{t,p}(a,b)}{T(a,b)}&=\ln\frac{Q_{t,p}(a,b)}{A(a,b)}-\ln\frac{T(a,b)}{A(a,b)}\\&=p\ln\bigl[1+(1-2t)^2x^2\bigr]-\ln x+\ln\operatorname{arcsinh}x.\end{aligned}$$

Thus, Theorem 1.1 follows from Lemma 2.3.

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